Topology Qualifying Examination: August 2022

Instructions: Solve **four** out of the **five** problems. Even if you attempt all five problems, specify which four you want graded.

You must justify your claims either by direct arguments or by referring to theorems you know.

Problem 1. In each of the following questions answer TRUE or FALSE. If your answer is TRUE then give a proof of the statement. If your answer is FALSE then give a counterexample to the statement.

(a) Let D^2 denote the 2-dimensional unit disc. There is no continuous function $f : D^2 \longrightarrow D^2$ without fixed points.

(b) Suppose that Y is an n-dimensional embedded submanifold of a simply connected $n+2$-dimensional manifold X. Then, the complement $X \setminus Y$ simply connected.

Problem 2. (a) Explain how to construct a CW complex with homology groups as below. Explain and justify your steps.

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & i = 1; \\ \mathbb{Z}, & i = 2; \\ \mathbb{Z}/3\mathbb{Z}, \oplus \mathbb{Z}/6\mathbb{Z}, & i = 3; \\ 0, & i \geq 4 \end{cases}$$

(b) Compute the co-homology groups $H^i(X;\mathbb{Z})$, $i \geq 0$, of the space X you constructed in (a).

(b) Compute the co-homology groups $H^i(X;\mathbb{Z}/5\mathbb{Z},)$, $i \geq 0$, of the space X you constructed in (a).

Problem 3. (a) Describe a CW-complex structure on the real projective space $\mathbb{R}P^n$ and use it to calculate the homology groups $H_i(\mathbb{R}P^n)$, for all i.

(b) For $k < n$ does $\mathbb{R}P^n$ deformation retract onto $\mathbb{R}P^k$? Justify your answer carefully.

Problem 4. (a) Let $\mathbf{R}P^2$ denote the real projective plane and let $M = S^1 \times S^1 \times S^1$. Prove that any continuous map

$$f : \mathbf{R}P^2 \times \mathbf{R}P^2 \longrightarrow M$$

is null-homotopic.

(b) Construct two *NON-equivalent* 3-sheeted covering spaces of $T := S^1 \vee S^1$. Explain why the spaces you constructed are non-equivalent.

Problem 5. Let $E \subset S^2$ be the equator of the 2-sphere and $f : E \longrightarrow S^1$ be an n-sheeted *covering map* of E onto S^1 (where $n > 0$). Form the quotient space X_n from the disjoint union $S^2 \coprod S^1$ by identifying each $x \in E \subset S^2$ with $f(x) \in S^1$.

(a) Calculate the fundamental group of X_n.

(b) The quotient map in part (a) gives a map $q : S^2 \longrightarrow X_n$. Calculate $H_2(X_n)$ and describe the map $q_* : H_2(S^2) \longrightarrow H_2(X_n)$, induced by q.

(c) Let $\widetilde{X}_n$ denote the universal cover of X_n. Show that for every n, there is a continuous map $S^2 \longrightarrow \widetilde{X}_n$ not homotopic to a constant map.